Notes: Conditional Probability

CS 3130/ECE 3530: Probability and Statistics for Engineers

August 31, 2017

Review of “English translation” for events:
- \( A \cap B \) = “both events \( A \) and \( B \) happen”
- \( A \cup B \) = “either event \( A \) or \( B \) (or both) happens”
- \( A^c \) = “event \( A \) does not happen”

Set Theory Rules: (try drawing Venn diagrams of these)
- Definition of set difference: \( A - B = A \cap B^c \) “event \( A \) happens, but \( B \) does not”
- Associative Law:
  \[(A \cup B) \cup C = A \cup (B \cup C)\]
  \[(A \cap B) \cap C = A \cap (B \cap C)\]
- Commutative Law:
  \[A \cup B = B \cup A\]
  \[A \cap B = B \cap A\]
- Distributive Law:
  \[(A \cup B) \cap C = (A \cap C) \cup (B \cap C)\]
  \[(A \cap B) \cup C = (A \cup C) \cap (B \cup C)\]
- DeMorgan’s Law:
  \[(A \cup B)^c = A^c \cap B^c\]
  \[(A \cap B)^c = A^c \cup B^c\]

Counting:
- Number of permutations of \( n \) items: \( n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \) (a.k.a. number of unique orderings)
- Number of ways to select \( k \) items out of \( n \) choices: \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \) (here order does not matter, just which \( k \) items you select)

Probability Rules:
- Equally likely outcomes: \( P(A) = \frac{|A|}{|\Omega|} \)
- Inclusion-Exclusion Rule: \( P(A \cup B) = P(A) + P(B) - P(A \cap B) \)
- Complement Rule: \( P(A^c) = 1 - P(A) \)
- Difference Rule: \( P(A - B) = P(A) - P(A \cap B) \)

Exercise: Try deriving these rules from the definition of a probability function. Draw a Venn diagram to convince yourself they work.
Conditional Probability:

\[ P(A \mid B) = \text{the probability of event } A \text{ given that we know } B \text{ happened} \]

Formula: \[ P(A \mid B) = \frac{P(A \cap B)}{P(B)} \]

Multiplication Rule:

\[ P(A \cap B) = P(A \mid B)P(B) \]

Tree diagrams to compute “two stage” probabilities (B = first stage, A = second stage):

1. First branch computes probability of first stage: \( P(B) \)
2. Second branch computes probability of second stage, given the first: \( P(A \mid B) \)
3. Multiply probabilities along a path to get final probabilities \( P(A \cap B) \)

Example: You are given two boxes with balls numbered 1 - 5. One box contains balls 1, 3, 5, and the other contains balls 2 and 4. You first pick a box at random, then pick a ball from that box at random. What is the probability that you pick a 2?

<table>
<thead>
<tr>
<th>Picking a Box</th>
<th>Picking a Ball</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td>1/3</td>
<td>1/2 \times 1/3 = 1/6</td>
</tr>
<tr>
<td></td>
<td>1/3</td>
<td>1/2 \times 1/3 = 1/6</td>
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<td>1/2</td>
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</tr>
<tr>
<td></td>
<td>1/2</td>
<td>1/2 \times 1/2 = 1/4</td>
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Sampling without replacement:
I have a box with 10 red balls and 10 green balls. I draw 2 balls from the box without replacing them. What is the probability that I get 2 red balls?

Let \( R_1 = \text{“first ball red”} \) and \( R_2 = \text{“second ball red”} \) and use product rule:

\[ P(R_1 \cap R_2) = P(R_1)P(R_2 \mid R_1) = \frac{1}{2} \times \frac{9}{19} = \frac{9}{38} \approx 0.24 \]

If I draw 3 balls without replacement, what is the probability that they are all red?

\[
P(R_1 \cap R_2 \cap R_3) = P(R_1 \cap R_2)P(R_3 \mid R_1 \cap R_2) = P(R_1)P(R_2 \mid R_1)P(R_3 \mid R_1 \cap R_2) = \frac{1}{2} \times \frac{9}{19} \times \frac{8}{18} = \frac{18}{171} \approx 0.11
\]

In-Class Problem: Exercise 3.2a