Sets

A **set** is a collection of unique objects.

Here “objects” can be concrete things (people in class, schools in PAC-12), or abstract things (numbers, colors).

Examples:

\[ A = \{3, 8, 31\} \]
\[ B = \{\text{apple, pear, orange, grape}\} \]
**Not** a valid set definition: \[ C = \{1, 2, 3, 4, 2\} \]
Sets

- Order in a set does not matter!
  \[\{1, 2, 3\} = \{3, 1, 2\} = \{1, 3, 2\}\]

- When \(x\) is an element of \(A\), we denote this by:
  \[x \in A.\]

- If \(x\) is not in a set \(A\), we denote this as:
  \[x \not\in A.\]

- The “empty” or “null” set has no elements:
  \[\emptyset = \{\}\]
Sample Spaces

Definition

A **sample space** is the set of all possible outcomes of an experiment. We’ll denote a sample space as \( \Omega \).

Examples:

- Coin flip: \( \Omega = \{H, T\} \)
- Roll a 6-sided die: \( \Omega = \{1, 2, 3, 4, 5, 6\} \)
- Pick a ball from a bucket of red/black balls: \( \Omega = \{R, B\} \)
Some Important Sets

- **Integers:**
  \[ \mathbb{Z} = \{ \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \} \]

- **Natural Numbers:**
  \[ \mathbb{N} = \{ 0, 1, 2, 3, \ldots \} \]

- **Real Numbers:**
  \[ \mathbb{R} = \text{“any number that can be written in decimal form”} \]
  \[ 5 \in \mathbb{R}, \quad 17.42 \in \mathbb{R}, \quad \pi = 3.14159 \ldots \in \mathbb{R} \]
Building Sets Using Conditionals

- Alternate way to define natural numbers:
  \[ \mathbb{N} = \{ x \in \mathbb{Z} : x \geq 0 \} \]

- Set of even integers:
  \[ \{ x \in \mathbb{Z} : x \text{ is divisible by } 2 \} \]

- Rationals:
  \[ \mathbb{Q} = \{ p/q : p, q \in \mathbb{Z}, q \neq 0 \} \]
Subsets

Definition
A set $A$ is a subset of another set $B$ if every element of $A$ is also an element of $B$, and we denote this as $A \subseteq B$.

Examples:

- $\{1, 9\} \subseteq \{1, 3, 9, 11\}$
- $\mathbb{Q} \subseteq \mathbb{R}$
- $\{\text{apple, pear}\} \not\subseteq \{\text{apple, orange, banana}\}$
- $\emptyset \subseteq A$ for any set $A$
Events

Definition

An event is a subset of a sample space.

Examples:

- You roll a die and get an even number:
  \[ \{2, 4, 6\} \subseteq \{1, 2, 3, 4, 5, 6\} \]

- You flip a coin and it comes up “heads”:
  \[ \{H\} \subseteq \{H, T\} \]

- Your code takes longer than 5 seconds to run:
  \[ (5, \infty) \subseteq \mathbb{R} \]
Set Operations: Union

**Definition**

The **union** of two sets $A$ and $B$, denoted $A \cup B$ is the set of all elements in either $A$ or $B$ (or both).

When $A$ and $B$ are events, $A \cup B$ means that event $A$ or event $B$ happens (or both).

Example:

$A = \{1, 3, 5\}$  “an odd roll”

$B = \{1, 2, 3\}$  “a roll of 3 or less”

$A \cup B = \{1, 2, 3, 5\}$
Set Operations: Intersection

**Definition**

The **intersection** of two sets $A$ and $B$, denoted $A \cap B$ is the set of all elements in both $A$ and $B$.

When $A$ and $B$ are events, $A \cap B$ means that both event $A$ *and* event $B$ happen.

Example:

$A = \{1, 3, 5\}$  "an odd roll"

$B = \{1, 2, 3\}$  "a roll of 3 or less"

$A \cap B = \{1, 3\}$

Note: If $A \cap B = \emptyset$, we say $A$ and $B$ are **disjoint**.
Set Operations: Complement

Definition

The **complement** of a set \( A \subseteq \Omega \), denoted \( A^c \), is the set of all elements in \( \Omega \) that are not in \( A \).

When \( A \) is an event, \( A^c \) means that the event \( A \) does not happen.

Example:

\[
A = \{1, 3, 5\} \quad \text{“an odd roll”}
\]

\[
A^c = \{2, 4, 6\} \quad \text{“an even roll”}
\]
Set Operations: Difference

**Definition**

The **difference** of a set \( A \subseteq \Omega \) and a set \( B \subseteq \Omega \), denoted \( A - B \), is the set of all elements in \( \Omega \) that are in \( A \) and are not in \( B \).

Example:
\[
A = \{3, 4, 5, 6\}
\]
\[
B = \{3, 5\}
\]
\[
A - B = \{4, 6\}
\]

Note: \( A - B = A \cap B^c \)
DeMorgan’s Law

Complement of union or intersection:

\[(A \cup B)^c = A^c \cap B^c\]

\[(A \cap B)^c = A^c \cup B^c\]

What is the English translation for both sides of the equations above?
Exercises

Check whether the following statements are true or false. (Hint: you might use Venn diagrams.)

- $A - B \subseteq A$
- $(A - B)^c = A^c \cup B$
- $A \cup B \subseteq B$
- $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
A probability function on a finite sample space $\Omega$ assigns every event $A \subseteq \Omega$ a number in $[0, 1]$, such that

1. $P(\Omega) = 1$
2. $P(A \cup B) = P(A) + P(B)$ when $A \cap B = \emptyset$

$P(A)$ is the probability that event $A$ occurs.
Equally Likely Outcomes

The number of elements in a set $A$ is denoted $|A|$.

If $\Omega$ has a finite number of elements, and each is equally likely, then the probability function is given by

$$ P(A) = \frac{|A|}{|\Omega|} $$

Example: Rolling a 6-sided die

$\quad P(\{1\}) = \frac{1}{6}$

$\quad P(\{1, 2, 3\}) = \frac{1}{2}$
Repeated Experiments

If we do two runs of an experiment with sample space $\Omega$, then we get a new experiment with sample space

$$\Omega \times \Omega = \{(x, y) : x \in \Omega, y \in \Omega\}$$

The element $(x, y) \in \Omega \times \Omega$ is called an **ordered pair**.

Properties:
Order matters: $(1, 2) \neq (2, 1)$
Repeats are possible: $(1, 1) \in \mathbb{N} \times \mathbb{N}$
More Repeats

Repeating an experiment \( n \) times gives the sample space

\[
\Omega^n = \Omega \times \cdots \times \Omega \quad (n \text{ times})
\]

\[
= \left\{ (x_1, x_2, \ldots, x_n) : x_i \in \Omega \text{ for all } i \right\}
\]

The element \((x_1, x_2, \ldots, x_n)\) is called an \( n \text{-tuple} \).

If \(|\Omega| = k\), then \(|\Omega^n| = k^n\).
Probability Rules

Complement of an event $A$:

$$P(A^c) = 1 - P(A)$$

Union of two overlapping events $A \cap B \neq \emptyset$:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
You are picking a number out of a hat, which contains the numbers 1 through 100. What are the following events and their probabilities?

- The number has a single digit
- The number has two digits
- The number is a multiple of 4
- The number is not a multiple of 4
- The sum of the number’s digits is 5
Permutations

A permutation is an ordering of an \( n \)-tuple. For instance, the \( n \)-tuple \((1, 2, 3)\) has the following permutations:

\[(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)\]

The number of unique orderings of an \( n \)-tuple is \( n \) factorial:

\[
n! = n \times (n - 1) \times (n - 2) \times \cdots \times 2
\]

How many ways can you rearrange \((1, 2, 3, 4)\)?
The **binomial coefficient**, written as \( \binom{n}{k} \) and spoken as “\( n \) choose \( k \)”, is the number of ways you can select \( k \) items out of a list of \( n \) choices.

**Formula:**

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}
\]
Binomial Coefficient or “$n$ choose $k$”

**Example:** You have cards numbered 1 through 10. If you pick five cards at random, what is the probability that you selected exactly the cards 1, 2, 3, 4, 5 (not necessarily in that order)?
Answer:

We’ll use the formula $P(A) = \frac{|A|}{|\Omega|}$.

There is only one combination that gives us cards 1,2,3,4,5, so $|A| = 1$.

The total number of possible 5 card selections is

$$|\Omega| = \binom{10}{5} = \frac{10!}{5!(10-5)!} = 252$$

So, finally the probability is

$$P(A) = \frac{|A|}{|\Omega|} = \frac{1}{252} \approx 0.00397 = 0.397\%$$