Probability

CS2100
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Monty Hall Problem

• 3 Doors
• Two “goats” and a “prize” (all expense paid trip to Logan)
• You pick a door
• Monty does not open it
  o Instead he opens one of the doors you did not pick, and there is a goat
• You can change your mind on the door or stick with your current door.
  o What is your best strategy?
  o What is the probability that you will win?
Probability

• De Méré – early 1600s
  ○ Gambling
  ○ How to split a pot from gambling

• De Morgan – early 1800s
  ○ “probability as meaning the state of mind with respect to an assertion, a coming even, or any other matter on which absolute knowledge does not exist”
Experiment

• Roll a pair of (6 sided) dice and record the sum of face-up numbers

• How do we represent the rolls so that the outcomes are equally likely?

• Represent sums:
  o \{2,3,4,5,6,7,8,9,10,11,12\}

• Represent individual events
  o (1,1), (1,2), (1,3), (1,4), (1,5), (1,6)
  o (2,1), (2,2), (2,3), (2,4), (2,5), (2,6)
  o (3,1), (3,2), (3,3), (3,4), (3,5), (3,6)
  o (4,1), (4,2), (4,3), (4,4), (4,5), (4,6)
  o (5,1), (5,2), (5,3), (5,4), (5,5), (5,6)
  o (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)
Mathematical Probability

- **Sample space** \((S)\) - set of outcomes of probabilistic experiment
  - Likelihood is known (default is equal)
- **A successful outcome**
  - A specified event happens
  - Some conditions are the outcome are met
  - Set of successful outcomes - \(E\)
- \(\text{Prob}(E) = |E|/|S|\)
Example

• What is the probability of rolling a sum of 10 on a pair of six sided dice?

• Draw two cards from deck of 52
  o what is the probability that the two cards have the same value (same face or number)

  o Strategy
    • Size of sample space (S) - events equally likely
    • Size of space of successes (E)
Examples

• Roll a die 3 times. What is the probability that a number comes up more than once?
  o Size of sample space:
    • \(|S| = 6 \times 6 \times 6\)
  o Size of E
    • Two strategies (one hard and one easy)
Constructive approach:
- How many ways are there to get two or more of any single number. E.g. “two 1’s”?
  - Getting exactly two is $C(3,2) \times 5 = 15$
  - There is one way to get three 1’s
  - Total of 16
  - There are six different options for the above so we have $6 \times 16 = 96$
- $\text{Prob}(E) = \frac{|E|}{|S|} = \frac{96}{216} = 0.44$
Example: Alternate Approach

• $|E| + |E'| = |S|

• Prob($E$) + Prob($E'$) = 1

• How many ways are there to get rolls with no repeated numbers ($E'$)?
  - $P(6,3)=120$
  - $\text{Prob}(E) = 1 - \text{Prob}(E') = 1 - \frac{120}{216} = 0.44$
Birthdays

• What is the probability that two or more people from a group of \( n \) have the same birthday?
Card Game

• 52 shuffled cards
• Go through deck and remove all adjacent pairs of same color (red or black)
• Repeat until
  o No adjacent pairs
  o Out of cards (win!)
Card Game (cont.)

• How to construct a winning deck:
  o Take one stack of 13 red and 13 black cards (stack 1) – any order.
  o Stack 2 is the remaining cards (13R and 13B) – any order.
  o Interleave these two stacks

• How do we know this is a winning deck?
  o Proof by induction on $n$, where $2n$ is total deck, stacks have same number of each color.
  o Base case: $n=0$. You win!
  o P($n$). The results are interleaved so a removal of an adjacent pair of same color removes one of that color from each stack.
  o Thus, removal of cards results in P($n-1$), which is a winning hand by inductive assumption
Card Game (cont.)

• How many ways are there to construct this winning hand from 52 cars (26R and 26B)?
• Choose 13 red and 13 black from the deck
  o $\binom{26}{13} \cdot \binom{26}{13}$
• How many total ways are there to divide the deck into two equal stacks?
  o $\binom{52}{26}$
• $\text{Prob}(E) = \frac{\binom{26}{13} \cdot \binom{26}{13}}{\binom{52}{26}} = 0.22$
Rules for Probability

• Two events or outcomes are **disjoint** (mutually exclusive) if they cannot both result from an experiment

• If $E_1$ and $E_2$ are mutually exclusive then

$$\text{Prob}(E_1 \text{ or } E_2) = \text{Prob}(E_1) + \text{Prob}(E_2)$$
Rule of Sums

• Backgammon: can move any piece distance equal to either of two dice.
• Need a five to “win”
• Need either a sum of five or a one or more fives on two dice.
• Mutually exclusive
• \( \text{Prob(win)} = \text{Prob}(E_1 \text{ or } E_2) = \text{Prob}(E_1) + \text{Prob}(E_2) \)
• \( \frac{11}{36} + \frac{4}{36} = \frac{15}{36} \)
Independent Events

- Cards with and without replacement
  - Draw a card—replace—draw a second card
  - Probability of two kings
  - Draw a card—draw a second card

- The outcomes of two experiments are **independent** if the outcome of one experiment does not impact the outcome of the other
Example

• A die is rolled and a card is drawn
• What is the probability that the die and card both show an even number?
  o Total number of possible outcomes
    • $6 \times 52$
  o Total number of successful outcomes
    • $3 \times 20$
  o $\text{Prob}(E) = \frac{(3 \times 20)}{(6 \times 52)} = \frac{3}{6} \times \frac{20}{52}$
Product Rule

- Two independent events $E_1$ and $E_2$
  - Outcomes of two independent experiments

$$\text{Prob}(E_1 \text{ and } E_2) = \text{Prob}(E_1) \cdot \text{Prob}(E_2)$$
Example

• Suppose you have unfair ("loaded") die, where the probability of 6 is 1/2, and the prob of all other numbers 1/10.

• What is the probability of rolling a 5 on the first toss and a 6 on the second toss?
Dependent Events

- Given events $E_1$ and $E_2$ we say that the *probability of $E_1$ given $E_2$, denoted by $\text{Prob}(E_1|E_2)$, is the probability that $E_1$ happens assuming that $E_2$ happens.*

- This is also called the conditional probability for $E_1$.

- If $E_1$ and $E_2$ are independent, then what is the conditional probability of $E_1$?
  - $\text{Prob}(E_1|E_2) = \text{Prob}(E_1)$
Committee Example

- 8 men and 12 women
- Committee of 3 chosen at random
- $E_1$ - at least one woman on committee, $E_2$ - at least one man on committee

Compute $\text{Prob}(E_1|E_2)$

- Compute compliment of man on committee (no man)
- $C(12,3)$
- Total committees are $C(20,3)$
- Cases with at least one man is $C(20,3) - C(12,3) = 920$
- How many of these have a list one woman?
- Add two cases: $C(8,1) \cdot C(12,2) + C(8,2) \cdot C(12,1) = 864$
- $\text{Prob}(E_1|E_2) = 864/920$
General Product Rule

• Given two events in an experiment(s), the probability of both \( E_1 \) and \( E_2 \) is

\[
\begin{align*}
\text{Prob}(E_1 \text{ and } E_2) &= \text{Prob}(E_1|E_2) \cdot \text{Prob}(E_2) \\
&= \text{Prob}(E_2|E_1) \cdot \text{Prob}(E_1)
\end{align*}
\]
“Marbles and Urns”

• Two marbles are chosen from a bag/urn containing 3 red, 5 white, and 8 green marbles

• What is the probability that both marbles are red?

• What is the probability that one is white and one is green?
Blackjack

• First card down and second card up
• Let T be down-card = value 10 (10,J,Q,K)
• Let A be up-card = Ace
• Compute
  o Prob(T|A)
  o Prob(A \land T)
Drug Testing

- When an athlete has used steroids the test is correct with probability 0.995
- When an athlete has not used steroids the test is correct with probability 0.98
- Assume 0.03 of all athletes use steroids
- What is the probability of $P$ and $S$
  - $P$ - positive test, $S$ - uses steroids
- What does “$P$ and $S$” mean?
- What is probability of “$P$ and not $S$”?
Bernoulli Trials

• Suppose a baseball player gets a hit with prob=1/3 every time at bat.

• What is the prob that he/she gets one exactly hit per four bats?
  o Analyze sequences with product rule
  o HNNN, N#NN, NNHN, NNNH
  o Product rule is \((1/3)(2/3)^3\) for each of these cases
  o They are mutually exclusive
  o Thus the OR operation invokes sumation
  o \(\text{Prob}(1 \text{ hit}) = 4 \cdot (1/3) \cdot (2/3)^3\)
Bernoulli Trials

• Given an event with prob \( p \), the prob of having \( k \) successes in \( n \) independent trials is

\[
B(p,n,k) = C(n,k) \cdot p^k \cdot (1-p)^{(n-k)}
\]

• Proof: Ordered list of length \( n \) with entries from \( \{S,F\} \). The number of lists with \( k \) S’s is \( C(n,k) \). Independence invokes the product rule to get \( p^k \cdot (1-p)^{(n-k)} \) for each case. Exclusivity invokes summation for the \( C(n,k) \) cases.
Example

• Fake coin gives heads with prob $4/7$
• What is the prob we get 8 or more heads in 10 tosses?
  - Cases: 8 heads, 9 heads, 10 heads
    - $C(10,8) \cdot (4/7)^8 \cdot (3/7)^2$
    - $C(10,9) \cdot (4/7)^9 \cdot (3/7)^1$
    - $C(10,10) \cdot (4/7)^{10} \cdot (3/7)^0$
  - Mutually exclusive so add...
    - 0.12555
Sports

- Barons win with prob 0.75
- What is the prob that they win a best of seven playoff series by a margin of 4-2?
- B – Barons, O – other
- Series must end with B
- There are \( \binom{5}{2} \) possible sequences of games
- Each sequence has probability \((0.75)^4 \cdot (0.25)^2\)
- Thus: \(10 \cdot (0.75)^4 \cdot (0.25)^2\)
World Series

• In 1992 announcers said that in the previous years the team that one the first of the best-of-seven series won the entire series 60% of the time. Is this surprising?

• Assume:
  o A wins first game
  o \( \text{prob(\text{win team A})} = \frac{1}{2} \)
  o independent Bernoulli trials
World Series

• Cases: A wins 4 game series, A wins 5 game series, A wins 6 game series, A wins 7 game series.

• A must win the first and last game (first game is a given)

• Combinations are:
  - 4G = \( C(2,2) \cdot (1/2)^3 \)
  - 5G = \( C(3,2) \cdot (1/2)^4 \)
  - 6G = \( C(4,2) \cdot (1/2)^5 \)
  - 7G = \( C(5,2) \cdot (1/2)^6 \)

• Mutually exclusive so:
  - \( \text{Prob}(A) = 4G + 5G + 6G + 7G = 66\% \)