The Discrete Fourier Transform

CS/BIOEN 4640: Image Processing Basics

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Given a *complex-valued* function $g : \mathbb{R} \rightarrow \mathbb{C}$, Fourier transform produces a function of frequency $\omega$:

$$
G(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) \cdot \left[ \cos(\omega x) - i \cdot \sin(\omega x) \right] \, dx
$$

$$
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) \cdot e^{-i\omega x} \, dx
$$
Review: The Dirac Delta

Definition

The **Dirac delta** or **impulse** is defined as

\[
\delta(x) = 0 \text{ for } x \neq 0, \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(x) \, dx = 1
\]

- The Dirac delta is *not* a function.
- It is undefined at \( x = 0 \).
- Has the property

\[
\int_{-\infty}^{\infty} f(x) \, \delta(x) \, dx = f(0) \quad \text{for any function } f
\]
The Fourier Transform of a Dirac Delta

\[ \mathcal{F}\{\delta(x)\}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x) \cdot e^{-i\omega x} \, dx \]

\[ = \frac{1}{\sqrt{2\pi}} e^{0} \]

\[ = \frac{1}{\sqrt{2\pi}} \]

- In other words, Fourier of a Dirac is constant
- So, it has equal response at all frequencies
Convolution with a Dirac Delta

Convolving a function $g(x)$ with a Dirac delta gives

$$(g \ast \delta)(x) = \int_{-\infty}^{\infty} g(y) \cdot \delta(y - x) \, dy$$

$$= g(x)$$

- So, convolving with Dirac is the identity operator
- Also can be seen in the Fourier domain:

$$\mathcal{F}\{g \ast \delta\} = \sqrt{2\pi} \mathcal{F}\{g\} \cdot \mathcal{F}\{\delta\} = \mathcal{F}\{g\}$$
The Comb

Definition

The **comb function** or **Shah function** is defined as an infinite sum of Dirac deltas:

\[
III(x) = \sum_{k=-\infty}^{\infty} \delta(x - k)
\]

Notice that just like the Dirac delta, the comb function is *not* a function.
Using the Comb to Sample

Given a continuous function $g(x)$, we can “sample” this function by multiplication by a comb:

\[
\bar{g}(x) = g(x) \cdot \Pi(x)
\]

\[
= \sum_{k=-\infty}^{\infty} g(k) \cdot \delta(x - k)
\]

Notice that $\bar{g}(x)$ is also not a function.
Using the Comb to Sample

\[ g(x) \]

\[ \overline{g}(x) \]

\[ \mathcal{H}(x) \]
The Discrete Fourier Transform

For a discrete signal $g(u)$, where $u = 0, 1, \ldots, M$, the discrete Fourier transform is given by

$$G(m) = \frac{1}{\sqrt{M}} \sum_{u=0}^{M-1} g(u) \cdot \left[ \cos \left( 2\pi \frac{mu}{M} \right) - i \cdot \sin \left( 2\pi \frac{mu}{M} \right) \right]$$

$$= \frac{1}{\sqrt{M}} \sum_{u=0}^{M-1} g(u) \cdot e^{-i2\pi \frac{mu}{M}}$$
Comparing Discrete and Continuous Fourier Transforms:

**Continuous Fourier Transform:**

\[
G(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) \cdot e^{-i\omega x} \, dx
\]

**Discrete Fourier Transform:**

\[
G(m) = \frac{1}{\sqrt{M}} \sum_{u=0}^{M-1} g(u) \cdot e^{-i 2\pi \frac{mu}{M}}
\]
The inverse DFT, analagous to the continuous case, just changes the sign in the exponent:

\[ g(u) = \frac{1}{\sqrt{M}} \sum_{m=0}^{M-1} G(m) \cdot e^{i \frac{2\pi mu}{M}} \]
The Fourier Transform of the Comb

Fourier transform of a comb is another comb:

$$\mathcal{F}\{\Pi(x)\} = \Pi\left(\frac{\omega}{2\pi}\right)$$

\[\begin{array}{c}
\tau = 1 \\
(a) \hspace{2cm} \text{Comb function: } \Pi_1(x) = \Pi(x) \\
-13 & -11 & -9 & -7 & -5 & -3 & -1 & 1 & 3 & 5 & 7 & 9 & 11 & 13 \\
\rightarrow & \rightarrow \tau = 1 \\
\end{array}\]

\[\begin{array}{c}
(b) \hspace{2cm} \text{Fourier transform: } \Pi\left(\frac{1}{2\pi}\omega\right) \\
-13 & -11 & -9 & -7 & -5 & -3 & -1 & 1 & 3 & 5 & 7 & 9 & 11 & 13 \\
\rightarrow \rightarrow \omega_0 = 2\pi \\
\end{array}\]
The Fourier Transform of the Comb

Spacing of comb in time domain is inversely related to spacing in frequency domain:

\[ \mathcal{F} \left\{ \Pi \left( \frac{x}{\tau} \right) \right\} = \tau \Pi \left( \frac{\tau \omega}{2\pi} \right) \]

\( \tau = 3 \)

Comb function: \( \Pi_3(x) = \Pi \left( \frac{x}{3} \right) \)

Fourier transform: \( 3 \Pi \left( \frac{3}{2\pi} \omega \right) \)

\( \omega_0 = \frac{2\pi}{3} \)
If $\bar{g}(x) = g(x) \cdot \text{III}(x)$, then

$$\bar{G}(\omega) = \sqrt{2\pi} G(\omega) \ast \text{III} \left( \frac{\omega}{2\pi} \right)$$

- We know convolution with $\delta$ is the identity
- So, this produces copies of the spectrum $G$ shifted to each peak of the comb.
Aliasing Caused by Sampling

(a) $G(\omega)$

(b) $\tilde{G}_1(\omega)$

(c) $\tilde{G}_2(\omega)$

$\omega_{\text{max}}$ $\omega_1$ $\omega_2$

aliasing
The Nyquist-Shannon Sampling Theorem

**Theorem**

A bandlimited continuous signal can be perfectly reconstructed from a set of uniformly-spaced samples if the sampling frequency is twice the bandwidth of the signal.