Properties of the Fourier Transform

CS/BIOEN 4640: Image Processing Basics

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A complex number can be given as an angle $\phi$ and a radius $r$

Think 2D polar coordinates

Exponential form:

$$re^{i\phi} = r \cos(\phi) + i (r \sin(\phi))$$
Given a *complex-valued* function $g : \mathbb{R} \to \mathbb{C}$, Fourier transform produces a function of frequency $\omega$:

$$G(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) \cdot \left[ \cos(\omega x) - i \cdot \sin(\omega x) \right] \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) \cdot e^{-i\omega x} \, dx$$
The Fourier transform is invertible. That is, given the Fourier transform $G(\omega)$ we can reconstruct the original function $g$ as

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(\omega) \cdot e^{i\omega x} d\omega$$

We use the notation:

- Fourier transform: $G = \mathcal{F}\{g\}$
- Inverse Fourier transform: $g = \mathcal{F}^{-1}\{G\}$
The Dirac Delta

**Definition**

The **Dirac delta** or **impulse** is defined as

\[
\delta(x) = 0 \text{ for } x \neq 0, \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(x) \, dx = 1
\]

- The Dirac delta is *not* a function.
- It is undefined at \( x = 0 \).
- Has the property

\[
\int_{-\infty}^{\infty} f(x) \delta(x) \, dx = f(0) \quad \text{for any function } f
\]
The Dirac Delta

Even though the Dirac delta is not a function, we will plot it like this:
A shifted impulse is given by $\delta(x - a)$

A scaled impulse is given by $s \cdot \delta(x)$

$$\int_{-\infty}^{\infty} s \cdot \delta(x) \, dx = s$$
Fourier Transform Pairs: Cosine

\[ g(x) = \cos(\omega_0 x) \]
\[ G(\omega) = \sqrt{\frac{\pi}{2}} \cdot (\delta(\omega + \omega_0) + \delta(\omega - \omega_0)) \]

(Here \( \omega_0 = 3 \))
Fourier Transform Pairs: Sine

\[ g(x) = \sin(\omega_0 x) \quad \quad G(\omega) = i \sqrt{\frac{\pi}{2}} \cdot (\delta(\omega + \omega_0) - \delta(\omega - \omega_0)) \]

(Here \( \omega_0 = 5 \))

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Fourier Transform Pairs: Gaussian

\[ g(x) = \frac{1}{\sigma} \exp\left(\frac{-x^2}{2\sigma^2}\right) \]

\[ G(\omega) = \exp\left(\frac{-\omega^2}{2 \cdot (1/\sigma^2)}\right) \]

(Here \( \sigma = 3 \))
Fourier Transform Pairs: Box

\[ g(x) = \begin{cases} 
1 & \text{if } |x| < b \\
0 & \text{otherwise}
\end{cases} \]

\[ G(\omega) = \frac{2 \sin(b \omega)}{\sqrt{2\pi} \omega} \]

(Here \( b = 2 \))
Properties: Linearity

Scaling:

\[ \mathcal{F}\{c \cdot g(x)\} = c \cdot G(\omega) \]

Addition:

\[ \mathcal{F}\{g_1(x) + g_2(x)\} = G_1(\omega) + G_2(\omega) \]
Properties for Real-Valued Functions

- If \( g \) is a real-valued function, then

\[
G(\omega) = G^*(-\omega)
\]

- If \( g \) is real-valued and even: \( g(x) = g(-x) \), then \( G(\omega) \) is real-valued and even

- If \( g \) is real-valued and odd: \( g(x) = -g(-x) \), then \( G(\omega) \) is purely imaginary and odd
Properties: Similarity

Stretching a signal horizontally leads to a shrinking of the Fourier spectrum:

\[ \mathcal{F}\{g(s \cdot x)\} = \frac{1}{|s|} \cdot G \left( \frac{\omega}{s} \right) \]

And vice versa, shrinking the signal causes a stretching in the Fourier spectrum
A horizontal shift of the signal results in a phase shift of the Fourier transform:

\[ \mathcal{F}\{g(x + d)\} = e^{-i\omega d} \cdot G(\omega) \]

- Notice magnitude \(|G(\omega)|\) stays the same
- This is a rotation in the complex plane by \(-\omega d\)
Convolution becomes multiplication in the Fourier domain:

\[ \mathcal{F}\{g(x) \ast h(x)\} = \sqrt{2\pi} \ G(\omega) \cdot H(\omega) \]

And vice versa, multiplication becomes convolution:

\[ \mathcal{F}\{g(x) \cdot h(x)\} = \frac{1}{\sqrt{2\pi}} \ G(\omega) \ast H(\omega) \]