Notes: Total Probability and Bayes’ Rule

CS 3130 / ECE 3530: Probability and Statistics for Engineers

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**Brain Teaser:** Birthday Paradox (See section 3.2 in book)

**Total Probability**

A set of events $B_1, B_2, \ldots, B_n$ is a **partition** of $\Omega$ if they are pairwise disjoint, that is, $B_i \cap B_j = \emptyset$ for any $i, j$, and if their union is equal to all of $\Omega$, that is, $B_1 \cup B_2 \cup \ldots \cup B_n = \Omega$.

Given a partition $B_1, B_2, \ldots, B_n$ of $\Omega$, the **law of total probability** states

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \ldots + P(A|B_n)P(B_n)$$

A common application of this rule is for any event $B$, where we will have $B, B^c$ forming a partition of $\Omega$. Here the total probability is just two terms:

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$$

**In-Class Problem:** You have two urns, one with 4 black balls and 3 white balls, the other with 2 black balls and 2 white balls. You pick one urn at random and then select a ball from the urn. What is the probability the ball is white?

**In-Class Problem:** You have a system with a main power supply and auxiliary power supply. The main power supply has a 10% chance of failure. If the main power supply is running, the auxiliary power supply also has a 10% chance of failure. But if the main supply fails, the auxiliary supply is more likely to be overloaded and has a 15% chance to fail. What is the probability that the auxiliary power will fail?

**Bayes’ Rule**

So far we’ve talked about conditional probabilities $P(A|B) = P(A \cap B) / P(B)$. Bayes’ Rule allows us to switch the order of $A$ and $B$, that is, to compute $P(B|A)$. Here’s how it works. First, use the multiplication rule two different times:

$$P(A \cap B) = P(A|B)P(B)$$

and

$$P(A \cap B) = P(B \cap A) = P(B|A)P(A)$$
This tells us that the two right hand sides above are equal:

\[ P(A|B)P(B) = P(B|A)P(A) \]

Solving for \( P(B|A) \), we get **Bayes’ Rule**:

\[ P(B|A) = \frac{P(A|B)P(B)}{P(A)} \]

**In-Class Problem:** In the urn problem above, if you picked a black ball, what is the probability that you had picked the first urn (the 4 black, 3 white urn)?

**In-Class Problem:** In the power supply problem above, if the auxiliary power fails, what is the probability that the main power also failed?