Notes: Conditional Probability

CS 3130 / ECE 3530: Probability and Statistics for Engineers

September 1, 2016

Probability Review

“English translation” for events:

- \( A \cap B = \) “both events \( A \) and \( B \) happen”
- \( A \cup B = \) “either event \( A \) or \( B \) (or both) happens”
- \( A^c = \) “event \( A \) does not happen”

Set Theory Rules: (try drawing Venn diagrams of these)

- Definition of set difference: \( A - B = A \cap B^c \) “event \( A \) happens, but \( B \) does not”
- Associative Law:
  \[
  (A \cup B) \cup C = A \cup (B \cup C) \\
  (A \cap B) \cap C = A \cap (B \cap C)
  \]
- Commutative Law:
  \[
  A \cup B = B \cup A \\
  A \cap B = B \cap A
  \]
- Distributive Law:
  \[
  (A \cup B) \cap C = (A \cap C) \cup (B \cap C) \\
  (A \cap B) \cup C = (A \cup C) \cap (B \cup C)
  \]
- DeMorgan’s Law:
  \[
  (A \cup B)^c = A^c \cap B^c \\
  (A \cap B)^c = A^c \cup B^c
  \]

Probability Rules:

- **Inclusion-Exclusion Rule**: \( P(A \cup B) = P(A) + P(B) - P(A \cap B) \)
- **Complement Rule**: \( P(A^c) = 1 - P(A) \)
- **Difference Rule**: \( P(A - B) = P(A) - P(A \cap B) \)

Exercise: Try deriving these rules from the definition of a probability function. Draw a Venn diagram to convince yourself they work.
Conditional Probability

Definition:
\[ P(A \mid B) = \text{the probability of event } A \text{ given that we know } B \text{ happened} \]

Formula:
\[ P(A \mid B) = \frac{P(A \cap B)}{P(B)} \]

Multiplication Rule:
\[ P(A \cap B) = P(A \mid B)P(B) \]

Tree diagrams to compute “two stage” probabilities (B = first stage, A = second stage):

1. First branch computes probability of first stage: \( P(B) \)
2. Second branch computes probability of second stage, given the first: \( P(A \mid B) \)
3. Multiply probabilities along a path to get final probabilities \( P(A \cap B) \)

Example: You are given two boxes with balls numbered 1 - 5. One box contains balls 1, 3, 5, and the other contains balls 2 and 4. You first pick a box at random, then pick a ball from that box at random. What is the probability that you pick a 2?

\[
\begin{array}{ccc}
\text{Picking a Box} & \text{Picking a Ball} & \text{Probability} \\
1/2 & 1/3 & 1/6 \\
1/2 & 1/3 & 1/6 \\
1/2 & 1/2 & 1/4 \\
1/2 & 1/2 & 1/4 \\
\end{array}
\]

Sampling without replacement:
I have a box with 10 red balls and 10 green balls. I draw 2 balls from the box without replacing them. What is the probability that I get 2 red balls?

Let \( R_1 = \text{“first ball red”} \) and \( R_2 = \text{“second ball red”} \) and use product rule:
\[ P(R_1 \cap R_2) = P(R_1)P(R_2 \mid R_1) = \frac{1}{2} \times \frac{9}{19} = \frac{9}{38} \approx 0.24 \]

If I draw 3 balls without replacement, what is the probability that they are all red?
\[ P(R_1 \cap R_2 \cap R_3) = P(R_1 \cap R_2)P(R_3 \mid R_1 \cap R_2) \quad \text{Multiplication rule for } (R_1 \cap R_2) \cap R_3 \\
= P(R_1)P(R_2 \mid R_1)P(R_3 \mid R_1 \cap R_2) \quad \text{Multiplication rule for } R_1 \cap R_2 \\
= \frac{1}{2} \times \frac{9}{19} \times \frac{8}{18} = \frac{18}{171} \approx 0.11 \]

In-Class Problem: Exercise 3.2a