Sample Spaces, Events, Probability

CS 3130 / ECE 3530:
Probability and Statistics for Engineers

August 25, 2016
Sets

**Definition**

A **set** is a collection of unique objects.

Here “objects” can be concrete things (people in class, schools in PAC-12), or abstract things (numbers, colors).

Examples:

\[
A = \{3, 8, 31\} \\
B = \{\text{apple, pear, orange, grape}\} \\
\textbf{Not} \text{ a valid set definition: } C = \{1, 2, 3, 4, 2\}
\]
Sets

- Order in a set does not matter!

\[ \{1, 2, 3\} = \{3, 1, 2\} = \{1, 3, 2\} \]

- When \( x \) is an element of \( A \), we denote this by:

\[ x \in A. \]

- If \( x \) is not in a set \( A \), we denote this as:

\[ x \not\in A. \]

- The “empty” or “null” set has no elements:

\[ \emptyset = \{ \} \]
Sample Spaces

**Definition**

A **sample space** is the set of all possible outcomes of an experiment. We’ll denote a sample space as $\Omega$.

Examples:

- **Coin flip**: $\Omega = \{H, T\}$
- **Roll a 6-sided die**: $\Omega = \{1, 2, 3, 4, 5, 6\}$
- **Pick a ball from a bucket of red/black balls**: $\Omega = \{R, B\}$
Some Important Sets

▶ Integers:

\[ \mathbb{Z} = \{ \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \} \]

▶ Natural Numbers:

\[ \mathbb{N} = \{ 0, 1, 2, 3, \ldots \} \]

▶ Real Numbers:

\[ \mathbb{R} = \text{“any number that can be written in decimal form”} \]

\[ 5 \in \mathbb{R}, \quad 17.42 \in \mathbb{R}, \quad \pi = 3.14159 \ldots \in \mathbb{R} \]
Building Sets Using Conditionals

- Alternate way to define natural numbers:
  \[ \mathbb{N} = \{ x \in \mathbb{Z} : x \geq 0 \} \]

- Set of even integers:
  \[ \{ x \in \mathbb{Z} : x \text{ is divisible by 2} \} \]

- Rationals:
  \[ \mathbb{Q} = \{ p/q : p, q \in \mathbb{Z}, q \neq 0 \} \]
Subsets

Definition
A set $A$ is a \textbf{subset} of another set $B$ if every element of $A$ is also an element of $B$, and we denote this as $A \subseteq B$.

Examples:
- $\{1, 9\} \subseteq \{1, 3, 9, 11\}$
- $\mathbb{Q} \subseteq \mathbb{R}$
- $\{\text{apple, pear}\} \not\subseteq \{\text{apple, orange, banana}\}$
- $\emptyset \subseteq A$ for any set $A$
Events

**Definition**

An **event** is a subset of a sample space.

Examples:

- You roll a die and get an even number: \( \{2, 4, 6\} \subseteq \{1, 2, 3, 4, 5, 6\} \)
- You flip a coin and it comes up “heads”: \( \{H\} \subseteq \{H, T\} \)
- Your code takes longer than 5 seconds to run: \((5, \infty) \subseteq \mathbb{R}\)
Set Operations: Union

Definition

The **union** of two sets \( A \) and \( B \), denoted \( A \cup B \) is the set of all elements in either \( A \) or \( B \) (or both).

When \( A \) and \( B \) are events, \( A \cup B \) means that event \( A \) or event \( B \) happens (or both).

Example:

\[
A = \{1, 3, 5\} \quad \text{“an odd roll”}
\]

\[
B = \{1, 2, 3\} \quad \text{“a roll of 3 or less”}
\]

\[
A \cup B = \{1, 2, 3, 5\}
\]
Set Operations: Intersection

**Definition**

The **intersection** of two sets $A$ and $B$, denoted $A \cap B$ is the set of all elements in both $A$ and $B$.

When $A$ and $B$ are events, $A \cap B$ means that both event $A$ and event $B$ happen.

Example:

$A = \{1, 3, 5\}$  “an odd roll”

$B = \{1, 2, 3\}$  “a roll of 3 or less”

$A \cap B = \{1, 3\}$

Note: If $A \cap B = \emptyset$, we say $A$ and $B$ are **disjoint**.
Set Operations: Complement

**Definition**

The **complement** of a set \( A \subseteq \Omega \), denoted \( A^c \), is the set of all elements in \( \Omega \) that are not in \( A \).

When \( A \) is an event, \( A^c \) means that the event \( A \) does not happen.

Example:

\[
A = \{1, 3, 5\} \quad \text{“an odd roll”}
\]

\[
A^c = \{2, 4, 6\} \quad \text{“an even roll”}
\]
Set Operations: Difference

Definition

The **difference** of a set $A \subseteq \Omega$ and a set $B \subseteq \Omega$, denoted $A - B$, is the set of all elements in $\Omega$ that are in $A$ and are not in $B$.

Example:

$A = \{3, 4, 5, 6\}$

$B = \{3, 5\}$

$A - B = \{4, 6\}$

Note: $A - B = A \cap B^c$
DeMorgan’s Law

Complement of union or intersection:

\[(A \cup B)^c = A^c \cap B^c\]

\[(A \cap B)^c = A^c \cup B^c\]

What is the English translation for both sides of the equations above?
Exercises

Check whether the following statements are true or false. (Hint: you might use Venn diagrams.)

- $A - B \subseteq A$
- $(A - B)^c = A^c \cup B$
- $A \cup B \subseteq B$
- $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
A probability function on a finite sample space $\Omega$ assigns every event $A \subseteq \Omega$ a number in $[0, 1]$, such that

1. $P(\Omega) = 1$
2. $P(A \cup B) = P(A) + P(B)$ when $A \cap B = \emptyset$

$P(A)$ is the probability that event $A$ occurs.
Equally Likely Outcomes

The number of elements in a set \( A \) is denoted \(|A|\).

If \( \Omega \) has a finite number of elements, and each is equally likely, then the probability function is given by

\[
P(A) = \frac{|A|}{|\Omega|}
\]

Example: Rolling a 6-sided die

- \( P(\{1\}) = \frac{1}{6} \)
- \( P(\{1, 2, 3\}) = \frac{1}{2} \)
Repeated Experiments

If we do two runs of an experiment with sample space \( \Omega \), then we get a new experiment with sample space

\[
\Omega \times \Omega = \{(x, y) : x \in \Omega, y \in \Omega\}
\]

The element \((x, y) \in \Omega \times \Omega\) is called an ordered pair.

Properties:
Order matters: \((1, 2) \neq (2, 1)\)
Repeats are possible: \((1, 1) \in \mathbb{N} \times \mathbb{N}\)
Repeating an experiment $n$ times gives the sample space

$$\Omega^n = \Omega \times \cdots \times \Omega \quad (n \text{ times})$$

$$= \{(x_1, x_2, \ldots, x_n) : x_i \in \Omega \text{ for all } i\}$$

The element $(x_1, x_2, \ldots, x_n)$ is called an $n$-tuple.

If $|\Omega| = k$, then $|\Omega^n| = k^n$. 

Probability Rules

Complement Rule:
Probability of the complement of an event $A$:

$$P(A^c) = 1 - P(A)$$

Inclusion-Exclusion Rule:
Union of two overlapping events $A \cap B \neq \emptyset$:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Note: it can be useful to re-arrange this equation!

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$
Exercise

You are picking a number out of a hat, which contains the numbers 1 through 100. What are the following events and their probabilities?

- The number has a single digit
- The number has two digits
- The number is a multiple of 4
- The number is not a multiple of 4
- The sum of the number’s digits is 5
A permutation is an ordering of an $n$-tuple. For instance, the $n$-tuple $(1, 2, 3)$ has the following permutations:

$$(1, 2, 3), (1, 3, 2), (2, 1, 3)$$

$$(2, 3, 1), (3, 1, 2), (3, 2, 1)$$

The number of unique orderings of an $n$-tuple is $n$ factorial:

$$n! = n \times (n - 1) \times (n - 2) \times \cdots \times 2$$

How many ways can you rearrange $(1, 2, 3, 4)$?